Cosmological consequences of the NonCommutative Geometry Spectral Action

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Cosmological consequences of the noncommutative geometry spectral action are presented. Neglecting the nonminimal coupling of the Higgs field to the curvature, background cosmology remains unchanged, and only the inhomogeneous perturbations will evolve differently from the equivalent classical system. However, considering the nonminimal coupling, corrections will be obtained even at the level of the background cosmologies. Finally, the Higgs field may act as an inflaton field, due to its nonminimal coupling with geometry.

PACS numbers:

I. INTRODUCTION

I will present cosmological consequences of the spectral action functional in the noncommutative geometry model of particle physics. The mathematical tools of noncommutative geometry [1] offer the means to describe geometry and the geometric properties of spaces, using the functions defined on the geometry, and the properties of functions defined on the spaces, respectively.

Much below the Planck scale, gravity can be safely considered as a classical theory. However, as energies approach the Planck scale, the quantum nature of space-time reveals itself and the Einstein-Hilbert action becomes an approximation. At such energy scales, all forces, including gravity, should be unified, so that all interactions correspond to just one underlying symmetry. As a result, one may expect that the nature of space-time would change at Planckian energies, in such a way so that one can indeed recover the (familiar) low energy picture of diffeomorphism, which governs General Relativity, and internal gauge symmetries, which govern gauge groups on which the Standard Model is based.

I will concentrate on a possible application of noncommutative geometry, namely that space-time at the Planck length, $l_{\rm Pl} = \sqrt{G\hbar/c^3}$, is described by a different form of geometry – whose choice will certainly effect the phenomenological consequences of the model – than the continuous four-dimensional manifold $\mathcal{M}$. I will work within a simple framework, based on the hypothesis that space-time is the product of a continuous manifold $\mathcal{M}$ by a discrete space $\mathcal{F}$; the easiest generalisation of a commutative space. Certainly, at Planckian energies the structure of space-time must be noncommutative in a nontrivial way, so that its low energy limit gives the simple product $\mathcal{M} \times \mathcal{F}$ we are considering here.

I will focus on a model proposed by Connes and collaborators [2], aiming at describing the Standard Model of particle physics coupled with gravity. Following the general noncommutative geometry approach, a physical system is contained in the algebra of functions (space-time) represented as operators on a Hilbert space (states), with the action and metric properties encoded in a generalised Dirac operator. The physical Lagrangian is computed from the asymptotic expansion in the energy scale Λ, of a natural action functionals, *spectral*

action, defined on noncommuatative spaces. The derived physical Lagrangian is determined from the geometric input, namely the choice of a finite dimensional algebra, thus the physical implications are closely dependent on the underlying chosen geometry.

The successful outcome of the noncommutative geometry model proposed by Connes and collaborators [2] lies in the fact that the physical Lagrangian, obtained from the asymptotic expansion of the spectral action, contains the full Standard Model Lagrangian with additional Majorana mass terms for the right-handed neutrinos, and gravitational and cosmological terms coupled to matter. The gravitational terms include: (i) the Einstein-Hilbert action with a cosmological term, (ii) a topological term which is related to the Euler characteristic of the space-time manifold, (iii) a conformal gravity term having the Weyl curvature tensor, and (iv) a conformal nonminimal coupling of the Higgs field to gravity. The existence of the two last contributions and the dependence of the coefficients of the gravitational terms on the Yukawa parameters of the particle physics content, diversify this model from the usual minimal coupling of gravity to matter. Thus, one expects some distinct consequences on early universe cosmology [3], and probably a more natural inflationary mechanism [4]. It is important to note that such features remain unique to this type of noncommutative geometry models.

This noncommutative geometry spectral action offers, to my opinion, a rich phenomenology (see, *e.g.* Refs. [3], [4], [5]), in both particle physics and cosmology, which may provide tests for the theory, as well as some insight for nontrivial noncommutative spaces near the Planck energy scale.

II.   NONCOMMUTATIVE GEOMETRY COUPLED TO GRAVITY

Consider the simplest extension of the smooth four-dimensional manifold, $\mathcal{M}$, by taking the product of it with a discrete noncommuting manifold $\mathcal{F}$ of dimension 6. This internal space has dimension 6 to allow fermions to be simultaneously Weyl and chiral (as within string theory), whilst it is discrete to avoid the infinite tower of massive particles that are produced in string theory. The noncommutative nature of $\mathcal{F}$ is given by a spectral triple, introduced by Connes [6] as an extension of the notion of Riemannian manifold to noncommutative geometry. The real spectral triple is given by the data $(\mathcal{A}, \mathcal{H}, D)$, defined as follows: $\mathcal{A}$ is an involution of operators on the Hilbert space $\mathcal{H}$, which is essentially the algebra of coordinates. It is required the algebra to be unital, in the sense that compact manifolds are considered. D is a linear self-adjoint operator acting on $\mathcal{H}$, such that all commutators $[D, a]$ are bounded for $a \in \mathcal{A}$, that gives the inverse line element.

By assuming that the algebra constructed in $\mathcal{M} \times \mathcal{F}$ is *symplectic-unitary*, the algebra $\mathcal{A}$ is restricted to be of the form

$$\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}) \ , \tag{2.1}$$

where $k = 2a$ and $\mathbb{H}$ is the algebra of quaternions. The choice $k = 4$ is the first value that produces the correct number of fermions in each generation (note however that the number of generations is an assumption in the theory), namely there are $k^2 = 16$ fermions in each of the three generations [7].

The Dirac operator D connects $\mathcal{M}$ and $\mathcal{F}$ via the spectral action functional on the spectral triple. It is defined as $\mathrm{Tr}\left(f\left(D/\Lambda\right)\right)$, where $f > 0$ is a cut-off (test) function and Λ is the cut-off energy scale. These three additional real parameters are physically related to the

coupling constants at unification, the gravitational constant and the cosmological constant. The asymptotic expression for the spectral action, for large energy Λ, is of the form

$$\mathrm{Tr}\left(f\left(\frac{D}{\Lambda}\right)\right) \sim \sum_{k \in \mathrm{DimSp}} f_k \Lambda^k \!\!\fint |D|^{-k} + f(0)\zeta_D(0) + \mathcal{O}(1) , \qquad (2.2)$$

where $f_k = \int_0^\infty f(v)v^{k-1}\mathrm{d}v$ are the momenta of the function f, the noncommutative integration is defined in terms of residues of zeta functions, and the sum is over points in the *dimension spectrum* of the spectral triple.

The basic symmetry for a noncommutative space $(\mathcal{A}, \mathcal{H}, D)$ is $\mathrm{Aut}(\mathcal{A})$, which includes both the diffeomorphism and internal symmetry transformations. The action functional considered to obtain the physical Lagrangian has a bosonic and a fermionic part. The bosonic action is a spectral function of the Dirac operator, while the fermionic one has the simple linear form $(\psi, D\psi)$, where ψ are spinors defined on the Hilbert space [8]. Applying this principle to the noncommutative geometry of the Standard Model leads to the Standard Model action coupled to Einstein and Weyl gravity plus higher order nonrenormalisable interactions suppressed by powers of the inverse of the mass scale of the theory.

To study the implications of this noncommutative approach coupled to gravity for the cosmological models of the early universe, one can concentrate just on the bosonic part of the action; the fermionic part is however crucial for the particle physics phenomenology of the model. Therefore, since for the time period we are concerned, namely between the unification and the electroweak epoch, the Higgs field is the most relevant one, we can just consider the part of the spectral action given by $\mathrm{Tr}\left(f\left(D_A/\Lambda\right)\right)$, where $D_A = D + A + \epsilon' J A J^{-1}$ is the Dirac operator with inner fluctuations given by the gauge potentials of the form $A = A^\dagger = \Sigma_k a_k[D, b_k]$, for elements $a_k, b_k \in \mathcal{A}$.

Writing the asymptotic expansion of the spectral action, a number of geometric parameters appear, which describe the possible choices of Dirac operators on the finite noncommutative space. These parameters correspond to the Yukawa parameters of the particle physics model and the Majorana terms for the right-handed neutrinos. The Yukawa parameters run with the renormalisation group equations of the particle physics model. Since running towards lower energies, implies that nonperturbative effects in the spectral action cannot be any longer safely neglected, any results based on the asymptotic expansion and on renormalisation group analysis can only hold for early universe cosmology. For later times, one should instead consider the full spectral action.

Finally, the empirical data taken as an input to the model are the following ones: (i) there are 16 chiral fermions in each of the 3 generations, (ii) the photon is massless, and (iii) there are Majorana mass terms for the neutrinos.

A. Phenomenology

The full Lagrangian of the Standard Model minimally coupled to gravity, can be written [2] as the asymptotic expansion of the spectral action on the product space-time $\mathcal{M} \times \mathcal{F}$. One can then extract phenomenological consequences. The relations between the gauge coupling constants have been found to coincide with those obtained in Grand Unified Theories. Namely, $g_2^2 = g_3^2 = (5/3)g_1^2$, and $\sin^2\theta_W = (3/8)$, a value also obtained in SU(5) and SO(10).

The model has a number of successful outcomes. It leads to an acceptable top quark mass of 179 GeV, neutrino mixing and see-saw mechanism to give very light left-handed

neutrinos are successfully identified, and correct representations of the fermions with respect to SU(3)×SU(2)×U(1) are derived [2].

Its drawbacks can be summarised as follows: (i) The unification of gauge couplings with each other and Newton constant do not meet at one point. (ii) The mass of the Higgs field is of the order of 170 GeV, a value which is recently ruled out experimentally. However, one should always keep in mind that higher order contributions to the Higgs potential may modify the prediction for the Higgs mass. (iii) No new particles besides those of the Standard Model are predicted and this may be problematic if new physics is discovered at the Large Hadron Collider. (iv) There is no explanation of the number of generations. (v) There are no constraints on the values of Yukawa couplings.

Before however drawing any *unfair* conclusions, one must keep in mind that we have considered the simplest generalisation of the commutative geometry, which certainly is an effective theory. As I have stated previously, at Planckian energies nontrivial generalisations of noncommutative spaces must be considered, which will certainly change the particle spectrum. In other words, before drawing any conclusions, one should firstly, include higher order corrections to the spectral action, to show gauge couplings unification, and thus get an accurate prediction for the Higgs mass, and secondly, find the noncommutative space whose limit is the product $\mathcal{M} \times \mathcal{F}$.

B. Cosmological consequences

In what follows, I will briefly summarise the results we have obtained on some cosmological consequences of this noncommutative geometry spectral action. In particular, I will first discuss corrections to Einstein's equations and then I will present a possible inflationary mechanism through the Higgs field.

1. Corrections to Einstein's equations

For the purpose of our work, namely extracting cosmological consequences of the noncommutative geometry coupled to gravity approach, we are only interested in the gravitational part of the action:

$$\mathcal{S}_{\mathrm{grav}} = \int \left(\frac{1}{2\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \tau_0 R^\star R^\star - \xi_0 R |\mathbf{H}|^2 \right) \sqrt{g} \mathrm{d}^4 x \ , \qquad (2.3)$$

where $\mathbf{H}$ is a rescaling $\mathbf{H} = (\sqrt{af_0}/\pi)\phi$ of the Higgs field ϕ to normalise the kinetic energy and $f_0 = f(0)$. The action of quaternions $\mathbb{H}$ can be represented in terms of Pauli matrices σ^α, namely $q = f_0 + \sum i f_\alpha \sigma^\alpha$, with a

$$a = \mathrm{Tr} \left(Y_{(\uparrow 1)}^\star Y_{(\uparrow 1)} + Y_{(\downarrow 1)}^\star Y_{(\downarrow 1)} + 3 \left(Y_{(\uparrow 3)}^\star Y_{(\uparrow 3)} + Y_{(\downarrow 3)}^\star Y_{(\downarrow 3)} \right) \right) \ ; \qquad (2.4)$$

the Y's are used to classify the action of the Dirac operator and give the fermion and lepton masses, as well as lepton mixing, in the asymptotic version of the spectral action. The

coupling constants in Eq. (2.3) are

$$\frac{1}{\kappa_0^2} = \frac{96 f_2 \Lambda^2 - f_0 c^2}{12\pi^2} \;,$$

$$\alpha_0 = -\frac{3 f_0}{10\pi^2} \;,\quad \tau_0 = \frac{11 f_0}{60\pi^2} \;,\quad \xi_0 = \frac{1}{12} \;; \tag{2.5}$$

Λ is the renormalisation cut-off and c is expressed in terms of Y_R which gives the Majorana mass matrix, $c = \mathrm{Tr}\,(Y_R^\star Y_R)$.

The first two terms in Eq. (2.3) give the Riemannian curvature with a contribution from the Weyl curvature; the second term is the action for conformal gravity. The third is a topological term integrating to the Euler characteristic of the manifold,

$$R^\star R^\star = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} R_{\mu\nu}^{\alpha\beta} R_{\rho\sigma}^{\gamma\delta} \;, \tag{2.6}$$

hence nondynamical. The fourth is the scalar mass term.

The equations of motion arising from Eq. (2.3) read [9]

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \alpha_0 \kappa_0^2 \delta\,(\Lambda) \left[2 C^{\mu\lambda\nu\kappa}_{\quad\ ;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa} \right] = \kappa_0^2 \delta\,(\Lambda)\, T^{\mu\nu}_{\mathrm{matter}} \;, \tag{2.7}$$

$$\text{where}\qquad \delta\,(\Lambda) \equiv [1 - 2\kappa_0^2 \xi_0 |\mathbf{H}|^2]^{-1} \;.$$

Neglecting the nonminimal coupling between the geometry and the Higgs field, i.e. setting $\phi = 0$, we have explicitly shown [3] that noncommutative corrections to Einstein's equations are present only for inhomogeneous and anisotropic space-times. This is the effect of the purely geometrical terms; the term $R^\star R^\star$ is topological and hence plays no rôle in dynamics, while the term $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ vanishes for homogeneous and isotropic metrics. As a result, one is left with just the Einstein-Hilbert term. One could have already anticipated this result, since any effects on the noncommutativity of space-time coordinates should vanish for homogeneous and isotropic backgrounds, having all their points equivalent.

Considering the nonminimal coupling of the Higgs field to the curvature, which cannot be neglected as we approach the early universe era, we have however obtained [3] corrections even for background cosmologies. This may lead to important phenomenological consequences [3], since the effect of the nonminimal coupling of the Higgs field to the curvature results to a larger Higgs mass, within the context of static geometries.

2. Higgs field driven inflation

The gravitational and Higgs part of the asymptotic expansion of the spectral action read [2]

$$\mathcal{S}_{\mathrm{grav}} = \int \left(\frac{1}{2\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \tau_0 R^\star R^\star + \gamma_0 - \xi_0 R |\mathbf{H}|^2 + \frac{1}{2} |D_\mu \mathbf{H}|^2 + V\,(|\mathbf{H}|) \right) \sqrt{g}\,\mathrm{d}^4 x \;, \tag{2.8}$$

where the potential $V\,(|\mathbf{H}|) = \lambda_0 |\mathbf{H}|^4 - \mu_0^2 |\mathbf{H}|^2$ is the standard Higgs potential, and the $\kappa_0^2, \alpha_0, \tau_0, \lambda_0, \mu_0$ are specified in terms of the cut-off energy scale Λ, the couplings and the coefficients f_k.

The above action, Eq. (2.8), implies that, in addition to the cosmological constant term γ_0, which we will neglect here, the geometry is nonminimally coupled to the Higgs field. Such modification to the Einstein-Hilbert action is indeed the one required so that the amplitude of the perturbations during Higgs field inflation gets reduced, allowing the Higgs field to satisfy simultaneously the Standard Model requirements, as well as those imposed to inflation from the WMAP5 measurements of the Cosmic Microwave Background temperature anisotropies.

Let me be more specific: It is a long standing proposal that the scalar field of the Standard Model, namely the Higgs field, could play the rôle of the inflaton. However, within the general relativistic cosmology, in order to get the correct amplitude of density perturbations, the Higgs mass would have to be some 11 orders of magnitude higher that the one required by particle physics. Luckily, this requirement does not apply in the context of the noncommutative geometry coupled to gravity approach.

To reduce the amplitude of the induced Higgs field perturbations, from λ_0 to λ_0/ξ_0^2, a nonminimal coupling, like the one that naturally appears in the context of the noncommutative geometry, has been postulated [10]. Thus, the Higgs field could play the rôle of the inflaton, provided inflation took place at a scale higher than the strong-weak unification scheme at 10^{17}GeV. Analysis of the running of couplings with the cut-off scale would determine the energy scale of inflation, a still open issue since it has been only studied neglecting the nonminimal coupling between the Higgs field and the curvature, which is indeed vital here.

Let me now emphasise that the Higgs driven inflation presented within this approach, does not suffer from the criticism against the *conventional* Higgs field driven inflationary models. It has been recently argued [11] that corrections to the semi-classical approximation, which are typically neglected in *standard* inflationary models, may no longer be neglected for such *exotic* (Higgs field driven) scenarios. This is certainly a valid criticism, which luckily does not apply in the noncommutative geometry context. The reason is simple: In *conventional* Higgs inflation there is a strong coupling, namely $\xi_0 \sim 10^4$, between the Higgs field and the Ricci curvature scalar. Thus, the effective theory ceases to be valid beyond a cut-off scale $m_{\rm Pl}/\xi_0$. However, one should know the Higgs potential profile for the field values relevant for inflation, namely $m_{\rm Pl}/\sqrt{\xi_0}$, and such values are much higher than the cut-off of the validity of the effective theory. This argument is clearly not applicable to the noncommutative Higgs driven inflation, where $\xi_0 = 1/12$.

III.   CONCLUSIONS

I have presented some cosmological consequences arising from the asymptotic expansion of the spectral action functional in the noncommutative geometry model of particle physics. This proposal has the potential of offering a concrete and fundamental theoretical context to build a cosmological model.

I have first discussed that neglecting the nonminimal coupling of the Higgs field to the curvature, noncommutative geometry corrections to Einstein's equations are present only for inhomogeneous and anisotropic cosmologies. However, considering the nonminimal coupling, there are corrections even for background cosmologies.

I have then presented how natural inflation may indeed occur as a consequence of the nonminimal coupling between geometry and the Higgs field. The term which has been introduced *ad hoc* in order to achieve a Higgs field driven inflation, arises naturally in the context of the noncommutative geometry coupled to gravity approach.

Certainly, many interesting questions, concerning both cosmological consequences as well as particle physics implications, remain to be addressed and we plan to inevstigate them in a future study.

IV.   ACKNOWLEDGEMENTS

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